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Ammann tilings: a classification and an application*

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Abstract

We propose a description of all Ammann tilings of a plane, a half-plane and a quadrant. Using the description we show that there are continuum different Ammann tilings of a plane and half-plane, but only three tilings of a quadrant. It is well known that all Ammann tilings are aperiodic. We show how one can use this fact to construct an aperiodic set of Wang tiles.

1 Introduction

1.1 Cutting a hexagon into similar parts

There is a non-convex hexagon with right angles that has the following property. It can be cut into two similar hexagons so that the scale factors are equal to ψ and ψ^2, where $\psi < 1$. As the square of the original hexagon should be equal to the sum of squares of the parts, the number ψ should satisfy the equation

$$\psi^4 + \psi^2 = 1.$$

That is, ψ^2 should be equal to the golden ratio $\varphi = \frac{\sqrt{5}-1}{2}$.

Such cut is shown in Fig. 1. The numbers on sides indicate their lengths, which are powers of ψ: a digit i means ψ^i. Using the equation $\psi^{n+2} + \psi^{n+4} = \psi^n$, it is easy to verify that the picture is consistent.

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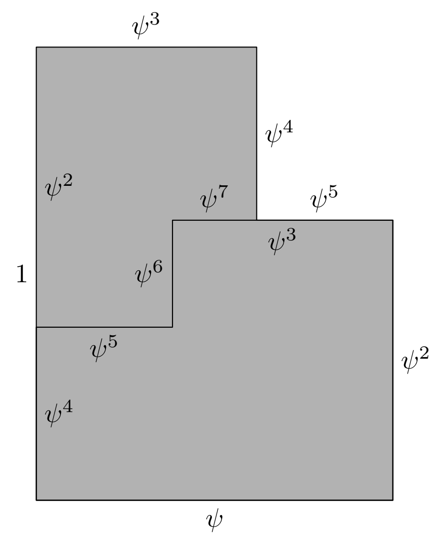


Figure 1: Cutting Ammann hexagon into similar parts.

We will call any hexagon that is similar to those on Fig. 1 *an Ammann hexagon*. The *size* of an Ammann hexagon is defined as the length of its largest side.

Are there other polygons that can be cut into two similar polygons with scale factors ψ and ψ^2? There is such right triangle: its altitude cuts it into two triangles that are similar to the original one; by adjusting the acute angle we can make the scale factors equal to ψ and ψ^2. The authors do not know any other such polygon.

1.2 Self-similar tilings

Defintion 1. In this paper, a *tiling* (more precisely, *a d-tiling*) is a non-empty set of Ammann hexagons that pair wise have no common interior points and each of them is either of size d (such hexagons are called *large*), or of size $d\psi$ (those are called *small*). A tiling T is a *tiling of a subset* A of the plane, or *tiles* A, if A equals the union of all hexagons in T, considered as subsets of the plane.

Defintion 2. A tiling T is called *periodic*, if there is a nonzero vector v (called a *period*) such the result of transition of every hexagon H in T by vector v belongs to T. Otherwise the tiling is called *aperiodic*.

For example, the tiling shown in Fig. 2 is a periodic tiling of the plane.

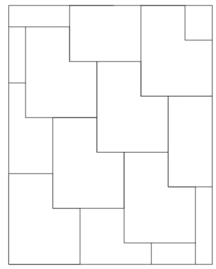
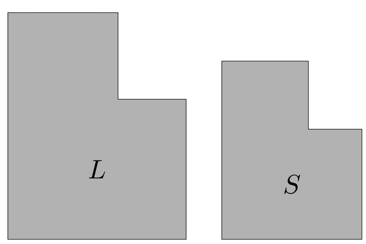


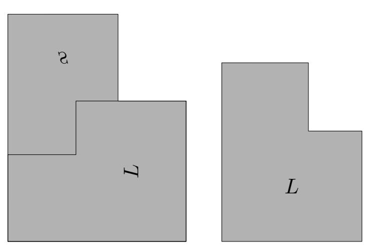
Figure 2: A periodic tiling of the plane.

In this paper we study special tilings of the plane, half-plane or a quadrant, which we call *Ammann tilings*. They are aperiodic and have other interesting features. Now we are going to define Ammann tilings.

Cut a given Ammann hexagon into two Ammann hexagons, as shown in Fig. 1. Then cut again the largest of the resulting hexagons into two Ammann hexagons. We obtain three parts: two large Ammann hexagons and one small Ammann hexagon. Again cut each of two large Ammann hexagons into two Ammann hexagons and so on. On each step we have a tiling consisting of Ammann hexagons of two sizes, which are denoted by L and S:



and cut all its large hexagons:



Small hexagons become large, and each large one is cut into a small one and a large one. All the resulting hexagons are similar to the original one, and the scale factor is a power of ψ. Tilings obtained in this way will be called *standard Ammann tilings*.

The operation used in this scheme will be called *refinement*, and the reverse operation will be called *coarsening*.

Defintion 3. The *refinement* of a d-tiling T is the $d\psi$-tiling obtained from T by cutting each large hexagon into two Ammann hexagons of sizes $d\psi$ and $s\psi^2$, as shown on Fig. 1, and keeping all small hexagons intact. If T is a refinement of S, then S is called *a coarsening of* T.

A $d\psi^n$-tiling obtained form a single Ammann hexagon H of size d by n successive refinements will be called a *standard tiling of level* n. A standard tiling of level 8 is shown in Fig. 3.

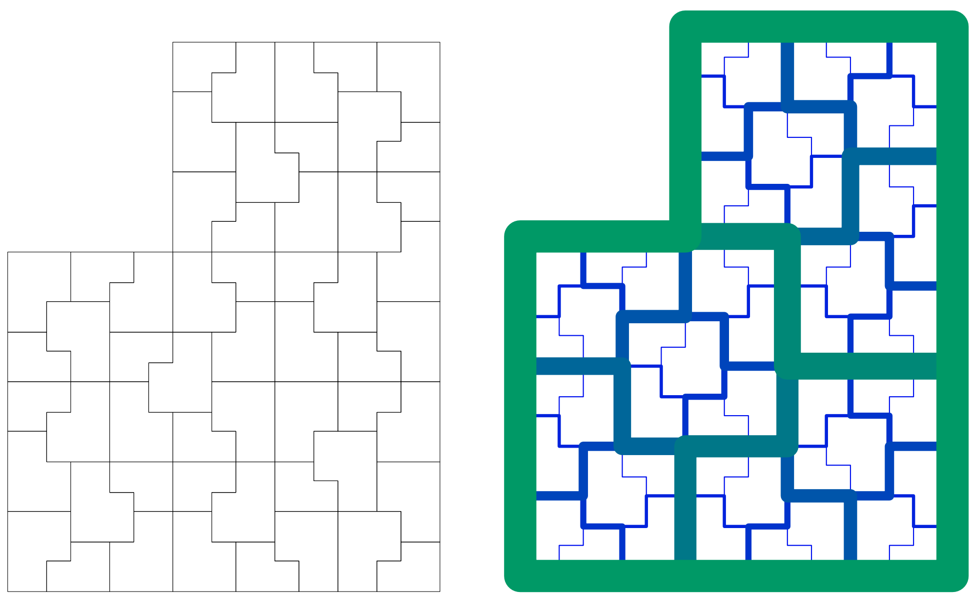


Figure 3: A standard tiling of level 8. In the right hand picture lines have different thickness to make the recursive structure visible.

Observe that for any $n \geqslant 2$, every standard d-tiling of level n is a disjoint union of a standard d-tiling of level $n-1$ and a standard d-tiling of level $n-2$, and every standard d-tiling of level 1 is a disjoint union of a standard d-tiling of level 0 and a d-tiling consisting of a single small hexagon. Therefore it is

convenient to call $\{H\}$, where H is a hexagon of size $d\psi$, *the standard d-tiling of level* -1 *of* H (so that the above rule be valid for all $n \geqslant 1$). A standard tiling is determined by the initial hexagon and by n or d. We will use both ways to identify a standard tiling.

In this paper we study tilings T of convex sets that look locally as standard tilings. What does it mean? It means that any finite subset of T is a subset of a standard tiling.

Defintion 4. A tiling T is an *Ammann tiling* if for all finite $W \subset T$ there is a standard Ammann tiling T' with $W \subset T'$.

Ammann tilings of convex subsets of the plane have several interesting properties. In this paper,

- we prove that if an Ammann tiling tiles a convex set then that set is a plane, half-plane or a quadrant;
- we prove that all Ammann tilings of convex sets are aperiodic;
- we present a way to construct continuum different Ammann tilings of the plane and half-plane, and prove that there are exactly 3 different Amman tilings of a quadrant,
- we present a description of all Ammann tilings of a plane, half-plane and a quadrant,
- we apply Ammann tilings to construct an aperiodic set of Wang tiles [3].

Notice that so far we have not even shown that there is an Ammann tiling of the entire plane. However, this is not a big deal. Indeed, we have already mentioned that every standard d-tiling T_n of level $n \geqslant 1$ is a union of a standard d-tiling T_{n-1} of level $n-1$ and a standard d-tiling T_{n-2} of level $n-2$. Given any standard standard d-tiling T_{n-1} of level $n-1$ it is not hard to find a standard d-tiling T_{n-2} of level $n-2$ so that together they form a standard tiling T_n of level n. In this way we can construct a sequence of tilings $T_0, T_1, T_2, \dots$ such that T_n is a standard tiling of level n and T_n is a subset of T_{n+1}. (Such sequence is uniquely defined by the choice of T_0.) Observe that the union $T = \bigcup_{n=0}^{\infty} T_n$ tiles the entire plane. On the other hand, by its construction T is an Ammann tiling.

2 Local rules

To fulfil the program we will give an equivalent more constructive definition of Ammann tilings of convex sets. We will do that be stipulating some its local properties and call *proper* all tilings that have those properties. Then we will prove that proper tilings satisfy all the statements we want to prove for Ammann tilings. In particular, we will describe their structure, which will imply that every proper tiling of a convex set is Ammann.

Call Ammann hexagons A, B *adjacent*, if they have no interior common points and their boundaries share a common segment (of positive length). It turns out that there is a local rule to attach Ammann hexagons to each other such that any tiling of a convex set obeying that rule is an Ammann tiling (and vice verse). We will define two such rules.

Local Rule 1. Call a pair (A, B) of adjacent Ammann hexagons *allowed* if there is a standard Ammann tiling that has both hexagons A and B. A tiling obeys this rule, if all pairs of adjacent hexagons in it are allowed. This is a weakening of the condition which defines Ammann tilings, therefore we will call such tilings *weakly Ammann tilings.*

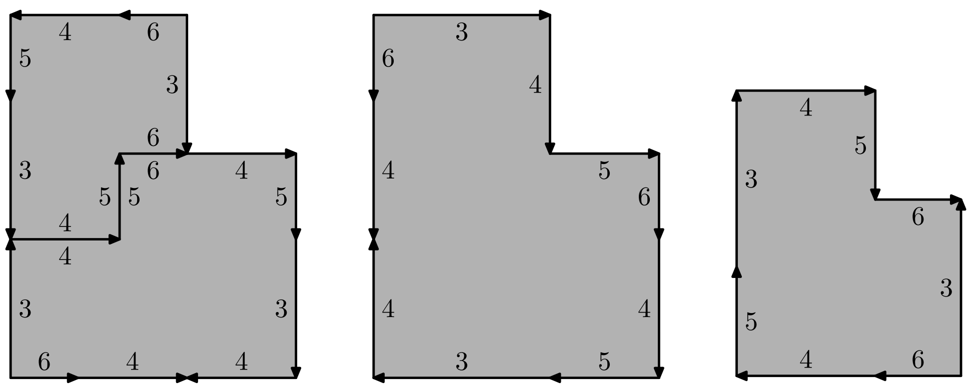
This rule is still not very constructive, as we do not list all allowed pairs.[1] Therefore we define another, more constructive, local rule.

Local rule 2. Colour in a given d-tiling the sides of large and small Ammann hexagons, as shown in Fig. 4(b) and 4(c). The sides of hexagons in this figure are divided into segments labeled by digits with arrows. Digits represent the colours and arrows identify orientations of segments. Digits correspond to the length of the segments (identify powers of ψ). This colouring of a given tiling will be called *standard.* A tiling is called *proper* if for every pair of adjacent hexagons H, G all adjacent points in H, G have the same colour and coloured segments have the same orientation.

Remark 1. For tilings of convex sets Local Rule 2 can be weakened without changing the class of proper tilings. Namely we can drop the requirement that adjacent segments of two different hexagons have the same orientation. Tiling obeying the weakened rule will be called *almost proper.* We will show in Appendix that all almost proper tilings of convex sets are proper (again just for fun).

Theorem 1. *Every standard tiling is proper.*

[1] Just for fun we present the list of all allowed pairs in Appendix, but do not prove that all pairs missing in the list are not allowed.



(a) (b) (c)

Figure 4: Coloring of the sides of Ammann hexagons

Proof. By induction: we will show that if a tiling is proper then so is its refinement. To verify this colour an Ammann hexagon of size d as a large hexagon (Fig. 4(b)). Then cut it into Ammann hexagons of sizes $d\psi$ and $d\psi^2$, as shown on Fig. 1. Change the coloured segments of the original hexagon using the following substitution:

$$\begin{aligned} \overrightarrow{6} &\to \overrightarrow{5} \\ \overrightarrow{5} &\to \overrightarrow{4} \\ \overrightarrow{4} &\to \overrightarrow{3} \\ \overrightarrow{3} &\to \overleftarrow{4}\,\overleftarrow{6} \end{aligned} \tag{1}$$

The reverse arrows in $\overleftarrow{4}$ and $\overleftarrow{6}$ mean that the transformation reverses the orientation. After that colour the cut by colours $\overrightarrow{4}$, $\overrightarrow{5}$, $\overrightarrow{6}$, as shown in Fig. 4(a). We will obtain two coloured hexagons shown in Fig. 4(a). They are coloured exactly as a large and small hexagons hexagons (Fig. 4(b) and 4(c)). One can verify this just by comparing Fig. 4(a), 4(b) and 4(c).

On the other hand, the same transformation of colours applied to any small hexagon (Fig. 4(c)) results in its colouring as a large hexagon (Fig. 4(b)).

Therefore refinement and subsequent re-colouring of a given standardly coloured tiling gets its standardly coloured refinement. As the transformation on colours we use is deterministic, this implies that a refinement of a proper tiling is proper as well. □

Remark 2. It is not surprising that the transformation used in the proof decreases the digits $6, 5, 4$ by 1: refinement corresponds to rescaling using the scale factor ψ. The digit 3 might be transformed to the digit 2, which corresponds to $4 + 6$ (more precisely, $\psi^2 = \psi^4 + \psi^6$).

Corollary 2. *Every weakly Ammann tiling is proper.*

Proof. Let T be an Ammann tiling. Let A, B be a pair of its adjacent hexagons. Being allowed, the pair (A, B) appears in a standard tiling, which is proper by Theorem 1. The latter implies that all adjacent points of A and B have the same colour and orientation of adjacent coloured segments in A and B is the same. $\square$

Thus the inclusions between the defined sets of tilings are the following: Ammann tilings $\subset$ Weakly Ammann tilings $\subset$ Proper tilings.

Both inclusions are proper for tilings of arbitrary sets. For instance, a proper tiling which is not weakly Ammann is shown in Fig. 5. It is not hard

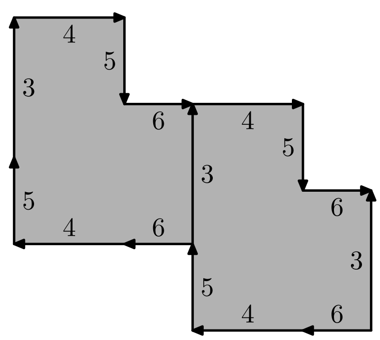


Figure 5: Proper tiling which is not weakly Ammann

to demonstrate that the shown pair of hexagons is not allowed. However, for tilings of convex sets, all the three families coincide. In other words:

Theorem 3. *Every proper tiling of a convex set is Ammann.*

This theorem is implied by the properties of proper tilings presented in the next section. The weaker statement asserting that every proper tiling (of a convex set) is weakly Ammann can be proved ad hoc by considering a large number of cases. Namely, one can consider all properly attached pairs (A, B) of Ammann hexagons (i.e., attached so that the adjacent coloured segment

have the same colour and orientation), and for each such pair either find that pair in Fig. 3 (it turns out that there are 15 such pairs, listed in Appendix) or show that the pair cannot occur in a proper tiling of a convex set (the latter can be verified by trying all the ways to fill the cavity in $A \cup B$, then trying to fill the cavity in the resulting figure and so on, and showing that all the ways are not successful).

3 The structure of proper tilings

We will prove here that every proper tiling (of a convex set) has a coarsening which is again proper. Then we will define infinite tilings that are very similar to standard tilings (in particular, they are Ammann tilings), called *infinite standard tilings*. Every infinite standard tiling tiles either the entire plane, or a half-plane or a quadrant. Then we will show that every proper tiling is either an infinite standard tiling or consists of two or four symmetrical infinite standard tilings. Infinite standard tilings have a clear structure. So we will get a clear understanding how proper tiling may look like. Using it we will prove that every proper tiling is Ammann.

3.1 Coarsening proper tilings

We introduce first some terminology and notation. Look at the hexagon in Fig. 1, call it A. The hexagon A is cut into a small and a large hexagons, denote them by S and L, respectively. We will call S the *son* of A and L the *daughter* of A. Similarly, S will be called the *sister* of L, while L will be called the *brother* of S. The hexagon A be called a *parent* of both L, S. Also, A will be called the *father* of L and the *mother* of S. An important property of these relations is the following: each Ammann hexagon has the unique son, daughter, brother, sister, father and mother. This is easy to verify by mere examining the shape of Ammann hexagons.

Lemma 4. *(a) For each d-tiling T there is at most one d/ψ-tiling S such that T is a refinement of S. (b) For each proper d-tiling T of a convex set there is such S (every proper tiling has the coarsening). (c) The coarsening of every proper tiling of a convex set is proper.*

Proof. (a) The first statement is almost obvious. Indeed, as we mentioned, every small hexagon (of size $d\psi$) has the unique brother. If the brother of

some small $H \in T$ is not in T then T has no coarsening. Otherwise the only tiling S whose refinement equals T is the tiling obtained from T by replacing each pair (brother, sister) by its common parent. (Some large hexagons in T might have no sisters in T, those hexagons become small hexagons in the coarsening of T.)

(b) Let T be a proper tiling of a convex set. We need to show that the brother of every small hexagon $H \in T$ belongs to T. Indeed, the cavity in H formed by arrows $\overrightarrow{5}$, $\overrightarrow{6}$ is somehow filled by another hexagon in T, call it G. Notice that only large Ammann hexagons have a right angle with arrows $\overrightarrow{5}$, $\overrightarrow{6}$ thus G is a large hexagon. There is only one such angle in every large hexagon and only one way to properly attach large hexagon to a small one to fill the gap, namely the way shown in Fig. 4(a). Thus G is necessarily the brother of H.

(c) Let T be a proper tiling of a convex set and T' its coarsening. As we have seen in the proof of Theorem 1, the standard colouring of T can be obtained from that of T' using the transformation (1). This transformation is injective except for the colour 4. This means that given the colour c and orientation of any point of hexagon in T we can reconstruct its colour in T' using the table (1), except the case $c = 4$. This shows that T' is proper except possibly for the points that have colour 4 in T.

The fate of points coloured 4, however, might depend on the hexagon. A segment coloured $\overrightarrow{4}$ in $H \in T$ becomes $\overrightarrow{5}$ in T' in either of the following cases: (a) it is not preceded by $\overrightarrow{6}$ in H or (b) is preceded by a segment $\overrightarrow{6}$ in H that is shared by the sister $K \in T$ of H (Fig. 4(a)). Otherwise the segment coloured $\overrightarrow{4}$ in $H \in T$ becomes a part of segment coloured $\overleftarrow{3}$ in T'. Thus we need to verify that the standard colouring of T has the following property: if a hexagon $H \in T$ has a pattern $\overrightarrow{6}\,\overrightarrow{4}$ and another hexagon $G \in T$ shares with H the arrow $\overrightarrow{4}$ from this pattern, then the arrow $\overrightarrow{4}$ is also preceded by $\overrightarrow{6}$ in G or that segment $\overrightarrow{6}$ is shared by the sister $K \in T$ of H. This is almost straightforward. Indeed, as T tiles a convex set, the arrow $\overrightarrow{6}$ must be shared by another hexagon $K \in T$ (different from H). We need to show that K and G coincide or K is the sister of H. Assume that $K \neq G$. Then they are separated by a line segment, as shown in Fig. 6. Thus the end of that arrow $\overrightarrow{6}$ is the angle of K. Notice that that might happen only if K is a small hexagon and $\overrightarrow{6}$ is an edge of its cavity. Thus K is the sister of H. $\square$

The tiling obtained from a tiling T by i coarsenings (if exists) will be

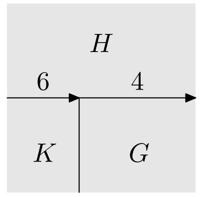


Figure 6: Hexagons H and K share the arrow 6

denoted by $T[i]$ and $T[-i]$ will denote the tiling obtained from a tiling T by i refinements.

Corollary 5. *Every proper tiling T of a convex set is aperiodic.*

Proof. Assume that T has a period v. Then v is also a period of the coarsening $T[1]$ of T (indeed, the refinement of $T' + v$ is equal to the tiling $T + v$, which equals T by the assumption; thus $T[1] + v$ and $T[1]$ have the same refinement and hence coincide by Lemma 4). Similarly, v is a period of the coarsening $T[2]$ of $T[1]$. And so on. Note that the coarsening increases the sizes of hexagons. Thus, on some step, v becomes smaller then the lengths of all sides of hexagons and we get a contradiction. □

Remark 3. Actually we have shown more. Assume that f is a mapping defined on the points of the plane such the distance between x and $f(x)$ is bounded by a constant. Assume that the image $f(H)$ of every hexagon H from a proper tiling T of a convex set is in T as well. Then $f(H) = H$ for all $H \in T$.

3.2 Infinite standard tilings

In this section we generalize the construction from Section 1 used to build a proper tiling of the plane.

Let $A_0, A_1, A_2, \dots$ be an infinite sequence of Ammann hexagon such that the size of A_0 is d or $d\psi$ and for all $n \geqslant 0$, A_{n+1} is the mother or the father of A_n. Let T_n stand for the standard d-tiling of A_n. Then $T_n \subset T_{n+1}$ for all n, except one case: if A_0 has size $d\psi$ and A_1 is its mother, then $T_0 = \{A_0\}$ and $T_1 = \{A_1\}$. Thus assume additionally that A_1 is the father of A_0 if A_0 has size $d\psi$. Then $\bigcup_{n=0}^{\infty} T_n$ is a d-tiling. We will call d-tilings of this form *infinite standard d-tilings.*

For example, if the size of A_0 is d and for all n, A_{n+1} is the father of A_n then $T_0, T_1, T_2, \dots$ is a sequence of standard d-tilings of levels $0, 1, 2, \dots$. The union of them is an Ammann tiling of the plane, used in Section 1 as an example of such tiling.

Every infinite standard tiling is Ammann and hence proper. The first natural question about infinite standard tilings is the following: what part of the plane do they tile? The second one is: are all infinite standard tilings the same (up to similarity)? We will show that there are infinitely many (even continuum) different infinite standard tilings and they might tile the entire plane, a half-plane and a quadrant.

3.3 When an infinite standard tiling tiles the entire plane?

It turns out that there is a simple description of infinite standard tilings that tile the entire plane. Infinite standard tilings are described as follows. Denote by $A \mapsto Al$, $A \mapsto As$ the mappings that map each Ammann hexagon to its father and mother, respectively. Every infinite standard d-tiling is identified by the hexagon A_0 and the sequence α of operations l, s such that A_{n+1} is obtained from A_n by the nth operation from α. Such tiling will be denoted by $\{A_0\}\alpha$.

Theorem 6. *(a) An infinite standard tiling $\{H\}\alpha$ does not tile the entire plane iff a tail of α consists of the blocks s and lsl. (b) In this case (when $\{H\}\alpha$ does not tile the plane) the tiling $\{H\}\alpha$ tiles a half-plane or a quadrant. More specifically, it tiles a quadrant iff a tail of α consist of alternating l and s (in other words, blocks s and lsl alternate: s–lsl–s–lsl–$\dots$).*

Proof. (a) Let us start with a simple observation:

Lemma. *If a part of the plane tiled by an infinite standard tiling $\{H\}\alpha$ interests a vertical line and intersects a horizontal line, then it includes the common points of the lines.*

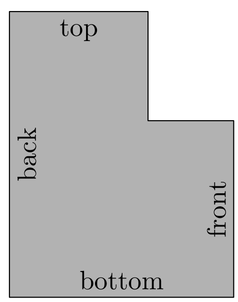
Proof. Let $T_0 = \{H\}$ and let T_i be obtained from T_{i-1} by the ith transformation from α. Let H_i stand for the Ammann hexagon tiled by T_i. For some i both lines intersect H_i. This does not imply yet that H_i contains their common point, as it might happen that it falls into the cavity of H_i. However in this case it falls into H_{i+1}. □

By the lemma every convex proper subset of the plane tiled by an infinite standard tiling does not intersect a vertical or a horizontal line. Hence it lies in one of the two half-planes defined by that line.

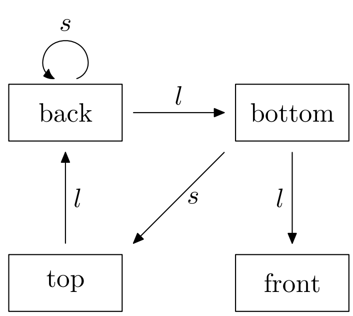
Consider the distance from the hexagons H_i to that line. As i is incremented by 1, the distance decreases at least by some non-negative d_i. The sequence $\{d_i\}$ is non-decreasing, hence starting from some i the distance does not change.

Therefore there is a line that touches almost all hexagons H_i. W.l.o.g. we may assume that it is a horizontal line and that all H_i lie above it. For almost all i the hexagon H_i has a side that lies on that line. Let us see what side of H_i that can be and how it changes as i increments.

We will view Ammann hexagons as small chair having the top, the back, the front, and the bottom:



The transformations l and s change the side of H_i that lies on the line as follows:



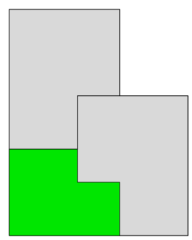
This fact is easy to verify by observing the cut in Fig. 1. If a transition is absent in this table, then that transition is impossible — the new hexagon would cross the line.

Thus, the "back" of H_i should infinitely many times lie on the line. Between two such i's only the transition s or a sequence of transitions lsl may

occur. This completes the 'only if' part of statement (a).

Conversely, assume that a tail of α consists of blocks s and lsl. W.l.o.g. we may assume that α itself consists of blocks s and lsl. Consider the line passing through the "back" of the hexagon H. Then all hexagons H_i lie in the same half-plane as H does. Hence the tiling $\{H\}\alpha$ tiles at most a half-plane.

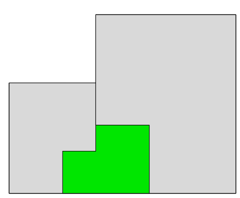
(b) Let us prove that if $\alpha = u\beta$ where β consists of the alternating transformations s and l, then $\{H\}\alpha$ tiles a quadrant. Assume first that u is empty. The mapping sl transforms the small green (grey in the black and white image) hexagon into a large hexagon that is inscribed in the same quadrant.



Therefore infinite number of applications of the transformation sl fills up the quadrant but not more. If u is not empty, then the same arguments apply to the hexagon Hu.

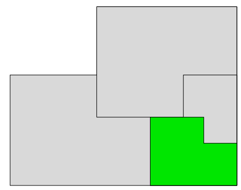
Assume now that a tail of α consists of the blocks s and lsl but they *do not* alternate. That is, the tail has infinitely many occurrences of ss or $lsllsl$.

Transformation s maps a hexagon that lies on a line on its back to a larger hexagon that also lies on the same line on its back. The second application of s increases the part of the hexagon that belongs to the line in the other direction.



Thus if a tail of α has infinitely many occurrences of ss (and consists of blocks s and lsl) then its application to an initial hexagons fills up a half-plane.

Similar arguments apply when α has infinitely many of occurrences of $lsllsl$. The mapping lsl also maps a hexagon that is attached to a line by its back to a larger hexagon that is again attached to the same line by its back.



Thus the double application of lsl increase the area of attachment in both directions. $\square$

3.4 Decomposing a proper tiling into a union of infinite standard tilings

Let T be a standard d-tiling of a hexagon A and $H \in T$. Then A can be obtained from H by a unique sequence u of transformations l, s. (For example, let H denote the hexagon in the bottom left corner of the tiling T shown on Fig. 3. Then the sequence is $lslsll$.) This is easily shown by induction on the level n of T. If $n = -1, 0$, the sequence is empty (in this case $A = H$ and $T = \{H\}$). Otherwise the last letter of u is necessarily l, if H is covered by the son of A, and is necessarily s, if it is covered by the daughter of A. In the first case we can apply the induction hypothesis the tiling of level $n-1$ of the son of A. In the second case we can apply the induction hypothesis the tiling of level $n-2$ of the daughter of A. This unique sequence u will be called the *address* of H in the standard d-tiling of A.

We want to define a similar notion for proper tilings T of convex sets. In this case the address is an infinite sequence and is defined as follows. Let T be a proper d-tiling of a convex set and $H \in T$. Let i be a natural number and A be a hexagon in $T[i]$. Then T is a disjoint union of standard d-tilings of A over all hexagons A in $T[i]$. Thus there is a unique $A \in T[i]$ such that H belongs to the standard d-tiling of A. And hence there is a unique u_i such that $Hu_i \in T[i]$. For all i the sequence u_i is a prefix of u_{i+1}. Indeed, if Hu_{i+1} is a small hexagon in $T[i+1]$, then Hu_{i+1} is in $T[i]$ as well and hence $u_i = u_{i+1}$. Otherwise Hu_i is a son or a daughter of Hu_{i+1} and hence u_{i+1} is obtained from u_i by appending l or s. Thus u_i's are prefixes of a unique infinite sequence α, which is called the *address* of H in T. Notice that if α is the address of α in T then the tiling $T = \{H\}\alpha$ is a subset of T.

Lemma 7. *If α is the address of H in an infinite standard tiling T then $T = \{H\}\alpha$.*

Proof. Being standard, T equals $\{G\}\beta$ for some infinite sequence of transformations β and some Ammann hexagon G. As H belongs to T, there are u, β' such that $\beta = u\beta'$ and H is in the standard d-tiling of Gu. Let v be the address of H in that d-tiling, that is, $Hv = Gu$. Thus the address of H in T equals $v\beta'$ and $T = \{G\}u\beta' = \{H\}v\beta' = \{H\}\alpha$. □

Theorem 8. *Each proper tiling T of a convex set is a union of one, two or four disjoint infinite standard tilings. Such decomposition is unique.*

Proof. Let H be a hexagon from T and let α be its address in T. If $\{H\}\alpha$ coincides with T then T is standard. Otherwise pick any hexagon G in $T \setminus \{H\}\alpha$ and let β be its address in T. The standard tiling $\{G\}\beta$ generated by G is disjoint with $\{H\}\alpha$. Indeed, assume that they share a hexagon K. Let γ_1, γ_2 be the addresses of K in $\{H\}\alpha, \{G\}\beta$ respectively. As both these tilings are subsets of T, the sequences γ_1, γ_2 are the addresses of K in T, as well. Thus $\gamma_1 = \gamma_2$ and by Lemma 7 the tiling $\{K\}\gamma_1$ coincides with both $\{H\}\alpha, \{G\}\beta$.

As each proper tiling covers at least a quadrant, in this way we can represent T as a disjoint union of one, two or four infinite standard tilings. Such representation is unique, as we have already shown that any two intersecting infinite standard sub-tilings of T coincide. □

In the next section we answer the question when infinite standard d-tilings are congruent.

3.5 When standard d-tilings are congruent

Obviously, finite standard d-tilings are congruent iff they have the same level. The level of the tiling $\{H\}\alpha$ can be computed as the sum of the level of $\{H\}$ (which is zero if H is large and -1 if H is small) and of the *weighted length* of the sequence α, which is defined as the number of occurrences of l plus the doubled number of occurrences of s. This answers the question for finite standard tilings.

For infinite standard tilings the answer is provided by the following

Lemma 9. *Infinite standard d-tilings $\{H\}\alpha$ and $\{G\}\beta$ are congruent iff $\alpha = u\gamma$ and $\beta = v\gamma$ for some u, v, γ such that the tilings $\{H\}u$ and $\{G\}v$ are congruent.*

Proof. The 'if' part is straightforward. The 'only if' part: assume that $\{H\}\alpha$ and $\{G\}\beta$ are congruent. W.l.o.g. we may assume that they coincide (otherwise apply an appropriate isometry to H). Let v be a prefix of β that is so large that Gv includes H, and let $\beta = v\gamma$. Then $Hu = Gv$ for some u. It remains to show that $\alpha = u\gamma$. This follows from the fact that both sequences α and $u\gamma$ are addresses of H in the tiling $\{H\}\alpha$:

$$\{H\}u\gamma = \{G\}v\gamma = \{G\}\beta = \{H\}\alpha.$$

□

The lemma implies that there are continuum (non-congruent) Ammann d-tilings. Moreover, there are continuum standard tilings of the plane and a half-plane. We will show now that there are only three proper tilings of a quadrant.

3.6 Proper tilings of a quadrant

From the previous section it follows that every proper tiling of a quadrant is standard. To show that there are only three such tilings we will use Lemma 9. Let $\{H\}\alpha$ be a standard tiling of a quadrant. W.l.o.g. we may assume that H is a large hexagon. Call sequences α, β of transformations l, s *equivalent* if $\alpha = u\gamma$ and $\beta = v\gamma$ for some u, v of the same weighted length. By Lemma 9 $\{H\}\alpha$ and $\{H\}\beta$ are congruent iff α and β are equivalent.

By Theorem 6 if $\{H\}\alpha$ (where H is a large hexagon) tiles a quadrant then α has a tail $slsl\ldots$. Let us show that there are three non-equivalent sequences α having such tail, namely

$$slsl\ldots, \qquad lsls\ldots, \qquad llsls\ldots \tag{2}$$

Indeed, in every sequence of transformations, we can replace s by ll, and back, without changing the equivalence class of the sequence. Therefore w.l.o.g. we may assume that $\alpha = uslsl\ldots$ where u consists of l's only, $u = ll\ldots l$. Replace in u every triple of consecutive l's by sl. In this way we get one of the sequences (2), depending on the residue of the length of u modulo 3.

On the other hand, it is easy to verify that the three above sequences are pair wise non-equivalent. On can see in Fig. 3 how look the corresponding tilings. The first one is obtained if we put the origin of the quadrant in the top right corner. To obtain the second tiling imagine that the bottom left

corner is the angle of the quadrant and for the third one — the bottom right corner.

Notice that three tilings have pairwise different sequences of colours on the sides of the quadrant (we will use this fact later). Indeed, for the first tiling (defined by the sequence $slsl\dots$) they start with

$$44664\dots, \qquad 3553\dots,$$

for the second tiling (defined by the sequence $lslsl\dots$) they start with

$$644446\dots, \qquad 335\dots,$$

and for the first tiling (defined by the sequence $llslsl\dots$) they start with

$$46644\dots, \qquad 533\dots.$$

3.7 Proper tilings of the plane and of a half-plane

There are proper tilings of the plane or a half-plane which are not standard. Indeed, we can pick any tiling of a half-plane and complete it using an axial symmetry to a tiling of the entire plane. The resulting tiling of the plane is non-standard, as it consists of at least two standard components. In a similar way, from any tiling of a quadrant, we can construct a non-standard tiling of a half-plane. However, it turns out that this is the only way to construct a non-standard tiling of a convex set. More specifically, every non-standard proper tiling of a convex set has an axe of symmetry, thus it consists of two symmetrical proper tilings.

Theorem 10. *(a) Every proper tiling of a half-plane is either standard, or consists of two axial symmetrical proper tilings of quadrants separated by its axe of symmetry. (b) Every proper tiling of the plane is either standard, or consists of two axial symmetrical proper tilings of half-planes separated by its axe of symmetry.*

Proof. (a) Let T be a proper tiling of a half-plane. As we have seen, it is either standard or consists of two disjoint standard tilings of quadrants.

Assume the second case. Then those quadrants must be separated by a ray that is orthogonal to the edge of the half-plane. As T is proper, the tilings of the quadrants have the same sequence of coloured segments on the separating ray. As we noticed in the last paragraph of Section 3.6, this

implies that the tilings are also the same, that is, the axial symmetry where the axe is the separating ray maps one tiling to the other.

(b) Let T be a proper tiling of a half-plane. Call the set of coloured oriented segments of hexagons in T lying on the edge of the half-plane the *shadow* of T. Each segment in the shadow is identified by the triple (start point, end point, colour). To prove the second statement we need to show that different tilings of a half-plane have different shadows.

Lemma 11. *Given the shadow of a proper tiling T of a half-plane we can reconstruct the tiling.*

Proof. We claim that the shadow of any proper tiling T of a half-plane is composed of blocks

$$\overrightarrow{6}\,\overrightarrow{4}\,,\quad \overleftarrow{4}\,\overleftarrow{6}\,,\quad \overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,,\quad \overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,,\quad \overrightarrow{5}\,\overrightarrow{3}\,,\quad \overleftarrow{3}\,\overleftarrow{5}\,,\quad \overrightarrow{5}\,\overrightarrow{3}\,\overleftarrow{3}\,,\quad \overrightarrow{3}\,\overleftarrow{3}\,\overleftarrow{5}\,.$$

Indeed, as we have seen in the proof of Theorem 6, only backs, fronts and tops of hexagons of a proper tiling can lie on the edge E of the half-plane tiled by it. In particular, this holds for the double coarsening $T[2]$ of T. A quick look at Fig. 4 reveals that the backs, fronts and tops of hexagons consists of blocks $\overrightarrow{6}\,\overrightarrow{4}$, $\overrightarrow{5}\,\overrightarrow{3}$, $\overrightarrow{3}$, $\overrightarrow{4}$. The double transformation (1) maps these blocks to $\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}$, $\overrightarrow{3}\,\overleftarrow{3}\,\overleftarrow{5}$, $\overleftarrow{3}\,\overleftarrow{5}$, $\overleftarrow{4}\,\overleftarrow{6}$, respectively.

The partition of the shadow into these blocks is unique (every $\overrightarrow{6}$ may be grouped only with the next $\overrightarrow{4}$ and every 'single' $\overrightarrow{4}$ can be grouped only with next $\overleftarrow{4}\,\overleftarrow{6}$; the same arguments apply to blocks of odd digits).

We claim that any of the blocks $\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}$, $\overrightarrow{3}\,\overleftarrow{3}\,\overleftarrow{5}$, $\overleftarrow{3}\,\overleftarrow{5}$, $\overleftarrow{4}\,\overleftarrow{6}$, identifies an Ammann hexagon or a pair of hexagons in T touching E, the edge of the half-plane. Indeed, $\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}$ can be only the back of a large hexagon, $\overrightarrow{6}\,\overrightarrow{4}$ can only be the bottom of a small hexagon, $\overrightarrow{5}\,\overrightarrow{3}\,\overleftarrow{3}$ can only be the back and top of a pair (sister, brother) and $\overrightarrow{5}\,\overrightarrow{3}$ can only be the bottom of a large hexagon. This is easy to verify looking at Fig. 4.

Thus we obtain a procedure to reconstruct a given tiling T from its shadow in the strip of width $d\psi^2$ near E. Using this procedure we can actually find T in any strip along E. Indeed to reconstruct T in the strip of width $d\psi^{2-i}$ near E first find the shadow of the tiling $T[i]$ obtained from T by i coarsenings (applying to the given shadow reverse transformations to (1)). Then apply the above procedure to reconstruct T_i in the strip of width $d\psi^{2-i}$ near E. Finally, apply i refinements to the obtained tiling. □

This completes the proof of Theorem 10. □

3.8 The proof of Theorem 3

Now we are able to prove Theorem 3. Let a proper tiling T of a convex set be given and let W be a finite subset of T. We have to show that W is a subset of a standard tiling.

If W is included in one standard component of T then we are done. Otherwise, T is a union of $k = 2$ or $k = 4$ standard tilings $\{H_1\}\alpha_1, \dots, \{H_k\}\alpha_k$ where $H_1, \dots, H_k$ are axial symmetrical. As the tilings are axial symmetrical, the sequences $\alpha_1, \dots, \alpha_k$ coincide. Let u be a prefix of α_1 such that the tiling $\{H_1\}u \cup \dots \cup \{H_k\}u$ includes W. If $k = 2$ then w.l.o.g. we may assume that the hexagons $H_1 u$ and $H_2 u$ share the common back lying on the axe of symmetry (Fig. 7). If $k = 4$ then w.l.o.g. we may assume that

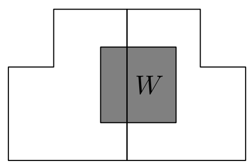


Figure 7: The tiling W is covered by two symmetrical hexagons.

the hexagons $H_1 u, H_2 u, H_3 u, H_4 u$ touch the axes of symmetry by their backs and bottoms (Fig. 8). The pair of hexagons shown in Fig. 7 can be easily

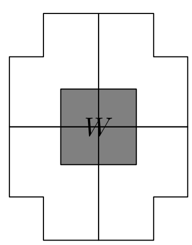


Figure 8: The tiling $W$is covered by four symmetrical hexagons.

found in Fig. 3, thus belongs to a standard tiling. The same applies to the quadruple of hexagons shown in Fig. 8. Refining that tiling several times we will obtain a tiling including W. This completes the proof of Theorem 3.

In the next question we study a very special question motivated by Lemma 11: which information in the shadow of a tiling of a half-plane is

essential to restore the tiling. The results in there are not used in the subsequent sections.

3.9 Recovering tilings of half-plane from its edge

A natural question regarding Lemma 11 is the following: how much information from the shadow we need to recover a tiling of a half-plane? To answer the question, define the *reduced shadow* of a tiling T of a half-plane as the set of vertices of hexagons in T that belong to the edge E of the half-plane (we have removed from the shadow the information about the division of sides of hexagons in coloured oriented segments).

Theorem 12. *Given the reduced shadow of a proper d-tiling of a half-plane we can restore the tiling.*

Proof. Let T be a proper d-tiling of a half-plane and let S be its shadow. Let E denote the edge of the half-plane. The shadow divides E into segments. Which lengths might those segments have? The top of a small hexagon cannot lie on E. Thus the shadow divides E into segments of lengths $d, d\psi, d\psi^2, d\psi^3$.

Let us denote the sides of lengths $d, d\psi, d\psi^2, d\psi^3$ by 0, 1, 2, 3, respectively. Let us orient them as shown in Fig. 9 (cf. Fig. 4 (b,c)):

$$\overrightarrow{0} = \overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}, \quad \overrightarrow{1} = \overleftarrow{3}\,\overleftarrow{5}, \quad \overrightarrow{2} = \overleftarrow{4}\,\overleftarrow{6}, \tag{3}$$

Lemma 13. *(a) The edge of any proper tiling of a half-plane includes either segments 0, 2, or segments 1, 3. In the first case only the following combinations can occur:*

$$\overrightarrow{2}\,\overrightarrow{0}, \quad \overleftarrow{0}\,\overleftarrow{2}, \quad \overleftarrow{2}\,\overrightarrow{2}, \quad \overleftarrow{0}\,\overrightarrow{0}, \quad \overrightarrow{0}\,\overleftarrow{0}.$$

In the second case only the following combinations can occur:

$$\overrightarrow{3}\,\overrightarrow{1}, \quad \overleftarrow{1}\,\overleftarrow{3}, \quad \overleftarrow{3}\,\overrightarrow{3}, \quad \overleftarrow{1}\,\overrightarrow{1}, \quad \overrightarrow{1}\,\overleftarrow{1}.$$

(b) Refinement transforms oriented segment as follows:

$$\begin{aligned} \overrightarrow{0} &\to \overleftarrow{1}\,\overleftarrow{3} \\ \overrightarrow{1} &\to \overrightarrow{0} \\ \overrightarrow{2} &\to \overrightarrow{1} \\ \overrightarrow{3} &\to \overrightarrow{2}. \end{aligned} \tag{4}$$

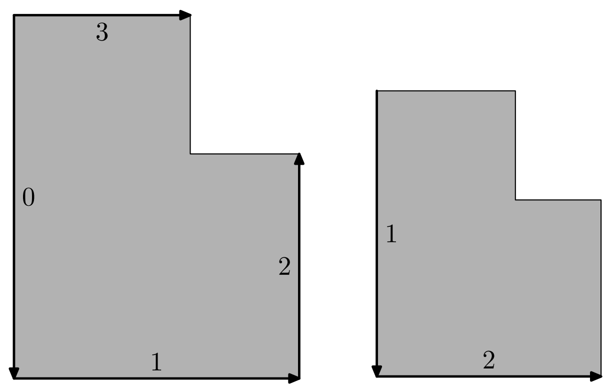


Figure 9: Orientation of sides on the edge of a half-plane

Proof. The statement (b) can be verified by transforming equations (3) using the table (1).

Let us prove (a). If a sequence satisfies (a) then the result of applying transformation (4) to it also satisfies (a). This can be verified by transforming all the allowed pairs of symbols (i.e. those listed in (a)) and checking that all the pairs of consecutive symbols in the resulting sequences are allowed. Therefore the sequence of $0, 1, 2, 3$ on the back of every finite standard tiling satisfies statement (a).

Let us first prove (a) for infinite standard tilings that tiles a half-plane. For such tilings, the sequence of $0, 1, 2, 3$ on the edge of the half-plane is equal to the limit of similar sequences for its finite parts. This proves (a) for infinite standard tilings.

Thus it remains to show (a) for tilings of a half-plane composed of two symmetrical infinite standard tilings of quadrants. Let E be the edge of such a tiling and let the axe of symmetry divide E into rays L_1, L_2. And let the bi-infinite sequence of oriented segments α along E be divided by that axe into α_1, α_2. We have already proved (a) for infinite standard tilings. Therefore both sequences α_1, α_2 satisfy (a). Thus we need only show that the pair (last segment of α_1)(first segment of α_2) is allowed. Recall that near the origin of the quadrant the tilings of quadrant look like the tiling shown in Fig. 3 near its three corners: the bottom left corner, the bottom right corner and the top right corner. So we obtain 6 possible pairs: $\overrightarrow{0}\overleftarrow{0}$, $\overleftarrow{1}\overrightarrow{1}$, $\overleftarrow{0}\overrightarrow{0}$, $\overleftarrow{3}\overrightarrow{3}$, $\overleftarrow{2}\overrightarrow{2}$, $\overrightarrow{1}\overleftarrow{1}$. All they are allowed. $\square$

Let us go on with the proof of the theorem. Every sequence satisfying the item (a) of the lemma has at most one pre-image under the transformation (4). For sequences composed of 1s and 3s this is obvious. And sequences composed of 0s and 2s can be uniquely decomposed into blocks of $\overrightarrow{2}\,\overrightarrow{0}$, $\overleftarrow{0}\,\overleftarrow{2}$, $\overleftarrow{0}$, $\overrightarrow{0}$.

The next observation is the following. If in a sequence satisfying (a) the orientation of arrows is dropped then we can restore it. Indeed, assume first that the sequence in question is composed of 0s and 2s. Then the orientation of 2s is restored by looking whether a given 2 is preceded or followed by 0 (and both cases cannot happen simultaneously, as the blocks $\overleftarrow{2}\,\overleftarrow{0}$ and $\overleftarrow{2}\,\overrightarrow{0}$ are not allowed). In this way we restore the orientation of all 2s and 0s following them. The orientation of the remaining 0s is restored using the fact that it alternates. For sequences composed of 1s and 3s, the observation is proved in an entirely similar way.

Given the reduced shadow of a proper tiling of a half plane we first restore the orientation of each segments of E, as described above. Using Equations (3) we restore the full shadow and then restore the tiling by Lemma 11. □

Is it possible to restore the tiling of a half-plane from a "half" of information in its shadow? More specifically, assume that a proper tiling T of a half-plane with the edge E is given. Assume that a ray R lying on E is given. Drop from the shadow S of T all segments that do not belong to the ray R. Is it always possible to recover the tiling T from the remaining part of the shadow? The answer is negative.

Lemma 14. *There are two different tilings T_1, T_2 of the same half-plane with an edge E and a ray R on E such that T_1 and T_2 have the same sequence of oriented coloured segments inside R.*

Proof. Let us reformulate the statement of the theorem in more combinatorial terms. The edge of any tiling of the half-plane is divided by the sides of its hexagons into oriented segments 0,1,2,3. Those segments form a bi-infinite sequence over the alphabet $\{\overleftarrow{0}, \overrightarrow{0}, \overleftarrow{1}, \overrightarrow{1}, \overleftarrow{2}, \overrightarrow{2}, \overleftarrow{3}, \overrightarrow{3}\}$. Call a bi-infinite sequences over this alphabet *admissible* if it appears on the edge of a proper tiling of a half-plane. We need to find admissible sequences α_1, α_2 such that $\alpha_1 = \gamma\beta_1$ and $\alpha_2 = \gamma\beta_2$ for some γ and different β_1, β_2.

To this end consider the tiling $\{G\}ssss\dots$ where G is a small hexagon. By Theorem 6 it tiles a half-plane. We will represent the sequence of oriented

segments along the edge of the tiling $\{G\}ssss\dots$ in the form $\beta^R w\beta$ where β and w have the following properties. Both β and w are sequence of segments $\overrightarrow{1}$, $\overleftarrow{1}$, $\overrightarrow{3}$, $\overleftarrow{3}$, where β is infinite and w is finite, β^R denotes the mirror image of β and w is different from its own mirror image w^R. Then we will let

$$\gamma = \beta^R, \quad \beta_1 = w\beta, \quad \beta_2 = w^R\beta.$$

The sequences β_1 and β_2 are different, since $w \neq w^R$.

Let

$$w = \overleftarrow{3}\,\overrightarrow{3}\,\overrightarrow{1}\,\overleftarrow{1}$$

and let

$$\beta = \overrightarrow{b_0}\,\overrightarrow{b_2}\,\overrightarrow{b_4}\,\overrightarrow{b_6}\dots$$

where $\overrightarrow{b_n}$ stands for the sequence of segments on the back of the standard tiling of level $n-1$ in the direction from top to bottom: $\overrightarrow{b_0} = \overrightarrow{1}$, $\overrightarrow{b_2} = \overleftarrow{1}\,\overleftarrow{3}$, and so on. Looking at Fig. 3 one can verify that for all even $n \geqslant 0$ the sequences $\overrightarrow{b_n}$ and $\overleftarrow{b_n}$ (where $\overleftarrow{b_n}$ denotes the mirror image of $\overrightarrow{b_n}$) satisfy the recurrence

$$\overrightarrow{b_{n+4}} = \overleftarrow{b_{n+2}}\,\overleftarrow{b_n}, \qquad \overleftarrow{b_{n+4}} = \overrightarrow{b_n}\,\overrightarrow{b_{n+2}}.$$

By construction we have

$$\beta^R w\beta = \dots\overleftarrow{b_6}\,\overleftarrow{b_4}\,\overleftarrow{b_2}\,\overleftarrow{b_0}\,\overleftarrow{3}\,\overrightarrow{3}\,\overrightarrow{1}\,\overleftarrow{1}\,\overrightarrow{b_0}\,\overrightarrow{b_2}\,\overrightarrow{b_4}\,\overrightarrow{b_6}\dots$$

Let α denote the sequence that appears at the edge of the tiling $\{G\}ssss\dots$. We need to show that $\beta = \alpha$. To this end notice, that α is the limit of the sequence of finite strings

$$\overrightarrow{b_0} \subset \overleftarrow{b_2} \subset \overrightarrow{b_4} \subset \overleftarrow{b_6} \subset \dots$$

defined by the above recurrence. More precisely, computing the limit, we assume that $\overrightarrow{b_{4n+4}}$ extends $\overleftarrow{b_{4n+2}}$ to the right (by appending $\overleftarrow{b_{4n}}$) and $\overleftarrow{b_{4n+6}}$ extends $\overrightarrow{b_{4n+4}}$ to the left (by prepending $\overrightarrow{b_{4n+2}}$). That is,

$$\alpha = \dots\overrightarrow{b_{10}}\,\overrightarrow{b_6}\,\overrightarrow{b_2}\,\overleftarrow{b_2}\,\overleftarrow{b_0}\,\overleftarrow{b_4}\,\overleftarrow{b_8}\dots$$

To show that $\beta = \alpha$ rewrite this this equality using the above recurrence:

$$\begin{aligned}\alpha &= \dots\overleftarrow{b_8}\,\overleftarrow{b_6}\,\overleftarrow{b_4}\,\overleftarrow{b_2}\,\overrightarrow{b_2}\,\overleftarrow{b_2}\,\overleftarrow{b_0}\,\overrightarrow{b_0}\,\overrightarrow{b_2}\,\overrightarrow{b_4}\,\overrightarrow{b_6}\dots\\ &= \dots\overleftarrow{b_8}\,\overleftarrow{b_6}\,\overleftarrow{b_4}\,\overleftarrow{b_2}\,\overleftarrow{b_0}\,w\,\overrightarrow{b_0}\,\overrightarrow{b_2}\,\overrightarrow{b_4}\,\overrightarrow{b_6}\dots = \beta^R w\beta. \quad \square\end{aligned}$$

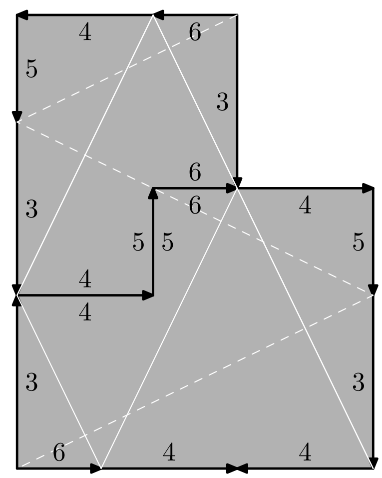


Figure 10: Solid and dotted lines

4 Ammann lines

Plot straight line segments (solid and dotted) inside the large and small Ammann hexagons, as shown in Fig. 10.

Theorem 15. *For every proper tiling of a convex part of the plane solid line segments form two families of parallel lines. The same holds for dotted lines.*

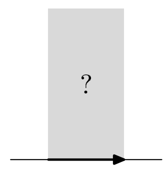
Proof. Let a proper tiling T of a convex set be given. Let us prove first that solid line segments in T form two families of parallel lines. Let us call a point of the plane *special* if it lies on a side of a hexagon in T and belongs to a solid line segment.

First, we need to verify that all lines form the same angle with the hexagon sides. This is easy to check looking at Fig. 10. Second, we have to show that every line segment is continued beyond its ends (which are special points) in the same direction, except for edges of the tiling.

Let us distinguish the following three types of special points:

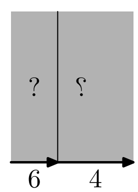
- points of degree 4 where right angles of four different hexagons meet;
- points of degree 3 where two right angles and a side of a hexagon meet;
- points of degree 2 which are inner points of sides of two attached hexagons.

Every special point is a starting point of an arrow and an ending point of another arrow. Notice that the arrow type determines uniquely what is drawn in a small neighbourhood of the arrow.



This is easy to verify by a look at Fig. 10.

This implies the solid line segments are symmetrical in a small neighbourhood of every arrow. In particular, in all points of degree 4 line segments continue each other (composition of two axial symmetries is a central symmetry). Every special point of degree 3 and 2 is an inner point of a side of a hexagon in T. Such points occur only where arrow 6 is continued by arrow 4 (this is easily checked on Fig. 10). The solid lines are symmetrical in every hexagon in a small neighbourhood of every such point:



Again we obtain a composition of two axial symmetries.

For dotted lines we cannot use similar arguments: they are not symmetrical in a neighbourhood of an arrow 6. To prove the statement for dotted lines notice that coarsening turns dotted lines into solid ones. Thus we can use Lemma 4 and the proven statement for solid lines. $\square$

4.1 Intervals between lines

Let a proper d-tiling of a convex set be given. As we have seen, solid Ammann lines for that tiling form two families. Each family has the following properties.

Theorem 16. *(a) Intervals between Ammann lines are of two lengths, l and $l\psi^2$ (for some l depending on d). There are no consecutive intervals of length $l\psi^2$. (b) Choose a direction along a line orthogonal to lines of the family and erase all Ammann lines next to which to the left there is a small interval.*

Then shift all remaining Ammann lines to the left by $l/2$. The resulting family is one of the two families of solid lines for the tiling which is obtained from the original one by two coarsenings.

Remark 4. Dotted lines are solid lines for the coarsening of the original tiling. Hence they also have properties (a) and (b).

Proof. By Theorem 10 every finite part of a proper tiling is included into a sufficiently large finite standard tiling. Therefore it suffices to prove the statements for finite standard tilings of even level.

(a) Let a standard tiling of even level be given. It tiles a part of the plane which is an Ammann hexagon. Draw on the back of that hexagon all special points. It suffices to show that intervals between them satisfy statement (a). We first show that those intervals are of lengths $2\alpha^4$ and $2\alpha^6$ only.

Lemma. *(a) The back of every standard tiling of even level consists of arrows* $\overrightarrow{6}, \overleftarrow{6}, \overrightarrow{4}, \overleftarrow{4}$ *and only the following pairs of consecutive arrows can occur in it:*

$$\overrightarrow{6}\,\overrightarrow{4}, \quad \overleftarrow{4}\,\overleftarrow{6}, \quad \overleftarrow{6}\,\overrightarrow{6}, \quad \overleftarrow{4}\,\overrightarrow{4}, \quad \overrightarrow{4}\,\overleftarrow{4}$$

(b) The sequence of arrows 4 and 6 on the back of standard tiling of even level $2n+2$ can be obtained from the analogous sequence of level $2n$ by the substitution

$$\overrightarrow{6} \to \overrightarrow{4}, \quad \overrightarrow{4} \to \overleftarrow{4}\,\overleftarrow{6}. \tag{5}$$

Proof. This lemma is very similar to Lemma 13 and is proven by the same arguments. To show (b) it is enough to notice that double refinement transforms the back of a standard tiling of level $2n+2$ into the back of standard tiling of level $2n$. On the other hand, double application of the mapping (1) yields the mapping (5).

The statement (a) is proven by induction. The base of induction: the back of the standard tiling of level 0 equals $\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}$ and thus satisfies (a). Inductive step: the mapping (5) respects (a). □

Notice that special points are exactly starting points of arrows $\overrightarrow{4}$. By the statement (a) the sequence of arrows on the back of every standard tiling of even level consists of blocks $\overrightarrow{4}\,\overleftarrow{4}$ and $\overleftarrow{6}\,\overrightarrow{6}$ (except possible for the first and the last arrow, which can be only $\overleftarrow{4}, \overrightarrow{6}$ and $\overrightarrow{4}, \overleftarrow{6}$, respectively). The length of the blocks $\overrightarrow{4}\,\overleftarrow{4}$ and $\overleftarrow{6}\,\overrightarrow{6}$ are $2\alpha^4, 2\alpha^6$, respectively, which proves the first part of statement (a). To the left and to the right of the block $\overleftarrow{6}\,\overrightarrow{6}$

only the block $\overrightarrow{4}\overleftarrow{4}$ can occur (or the first or last single arrow), as the pair $\overrightarrow{6}\overleftarrow{6}$ is not allowed. Hence there are no two consecutive intervals of length $2\alpha^6$ between special points.

(b) Erasing lines between a small interval and a large interval corresponds to erasing special points that happen to be the starting points of arrows $\overrightarrow{4}$ inside blocks $\overleftarrow{6}\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}$. Shifting to the left by a half of large interval corresponds to moving all special points from the beginnings of arrows 4 to their ends. After performing these two operations, special points cut the sequence of arrows on the back of a standard tiling into blocks $\overleftarrow{4}\overleftarrow{6}\overrightarrow{6}\overrightarrow{4}$ and $\overleftarrow{4}\overrightarrow{4}$. Double coarsening using (5) maps those blocks to blocks $\overrightarrow{4}\overleftarrow{4}$ and $\overleftarrow{6}\overrightarrow{6}$, respectively. Those blocks corresponds to the intervals between the lines of doubly coarsened tiling. $\square$

5 An aperiodic Wang tiles set

In this section we derive from the results of previous sections the existence of a set of Wang tiles that can tile the plane but only aperiodically. This is a well known fact, the first such set appeared in [2].

A *Wang tile* [3] is a square of size 1×1 whose sides have colours (all points of each side have the same colour). A tiling of a plane by Wang tiles is *proper* if every two adjacent sides of different tiles in it have the same colour. A set of Wang tiles is *aperiodic* if using tiles of that set we can properly tile the plane (each tile can be used infinitely many times, rotation of tiles is prohibited) and every proper tiling is aperiodic.

More formally, a tile set is a family F of functions from $\{N, W, S, E\}$ to $\{1, \dots, m\}$, where m stands for the number of colours, and N, W, S, E mean North, West, South and East and stand for the names of the sides of tiles. A tiling by tiles from F is a mapping U from $\mathbb{Z}\times\mathbb{Z}$ to F. It is *proper* if for all pairs (i, j) of integer numbers, $U(i, j+1)(S) = U(i, j)(N)$ and $U(i+1, j)(W) = U(i, j)(E)$. It is *periodic* if for some a, b and all i, j it holds $U(i+a, j+b) = U(i, j)$ and at least one of integers a, b is different from 0.

Theorem 17 ([2])**.** *There is an aperiodic set F of Wang tiles.*

The proof based on Ammann tilings. We will construct first a different thing. That A be a colouring of $\mathbb{Z}\times\mathbb{Z}$ in k colours, that is, a mapping from $\mathbb{Z}\times\mathbb{Z}$ to $\{1, 2, \dots, k\}$. Let n be a natural number. Look at A through windows of

size $n \times n$. Depending on the position of the window we can a see a picture, which is a mapping from $n \times n$ to $\{1, 2, \ldots, k\}$. Namely, through the window whose bottom left corner is located at (i, j) we can see the picture P defined by equation $P(x, y) = A(i+x, j+y))$. Let A_n stand for the set of all pictures we can see in A through windows of size $n \times n$.

Lemma. *There are a natural k and a mapping A from $\mathbb{Z}\times\mathbb{Z}$ to $\{1, 2, \ldots, k\}$ such that every mapping B from $\mathbb{Z} \times \mathbb{Z}$ to $\{1, 2, \ldots, k\}$ with $B_3 \subset A_3$ is aperiodic.*

Proof. Fix a proper tiling T of the plane by large and small Ammann hexagons. Plot all solid Ammann lines for that tiling. They divide the plane into parallelograms of four sizes, $2\alpha^4 \times 2\alpha^4$, $2\alpha^4 \times 2\alpha^6$, $2\alpha^6 \times 2\alpha^4$ and $2\alpha^6 \times 2\alpha^6$. Every parallelogram has some arrows inside it. Let us label every such arrow by two letters L, S indicating which hexagon is to the right and to the left of the arrow (the direction of arrows allows to distinguish the left hand and the right hand sides). It is not hard to see that there are only finitely many types of such parallelograms (two parallelograms with arrows are of the same type if they are congruent). Indeed, from Fig. 10 it is clear that solid lines divide hexagons in 5 parts. Counting isometric images we get at most $5\times4\times2 = 40$ parts. Those parts can be properly attached to each other in finitely ways to build a parallelogram of size at most $2\alpha^4 \times 2\alpha^4$.

Let k be equal to the number of different types of parallelograms. The parallelograms in the tiling T behave like Wang tiles: a side of one parallelogram is attached to a side of another one. Thus our tiling defines a mapping A from $\mathbb{Z} \times \mathbb{Z}$ to $\{1, \ldots, k\}$.

It remains to show that every other mapping B from $\mathbb{Z}\times\mathbb{Z}$ to $\{1, \ldots, k\}$ with $B_3 \subset A_3$ is aperiodic. To this end fix any such B. Replace on the plane, for all pairs (i, j) of integers, the unit square with the bottom left corner (i, j) by the parallelogram $B(i, j)$. We will obtain a tiling of the plane by parallelograms. Indeed, only parallelograms of the same horizontal size are vertically adjacent in A. Hence B has the same property (vertical adjacency can be seen even through windows of size 1×2). The same applies to horizontally adjacent parallelograms.

Arrows inside parallelograms of that tiling divide the plane into Ammann hexagons. Indeed, any anomaly could be observed inside a part consisting of 3×3 parallelograms (as every Ammann hexagon has at most two parallel solid line segments and thus is divided by them in at most 3 parts). This

tiling, call it T, is proper (we do not need prove this, as this follows from the construction of the tiling).

We have seen that every proper Ammann tiling of the plane is aperiodic. We will deduce from this that B is aperiodic as well. For the sake of contradiction assume that B is periodic: there is a non-zero shift g of $\mathbb{Z} \times \mathbb{Z}$ that preserves B. We would like to conclude from this that T is periodic as well. The shift g defines in a natural way a mapping $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$: a point (x, y) inside a parallelogram $B(i, j)$ is mapped by f to the same point inside $B(g(i, j))$. This mapping preserves T (that is, $f(H) \in T$ for all $H \in T$) and has no fixed points ($f(H) \neq H$ for all $H \in T$). Unfortunately, parallelograms are of different sizes. By this reason f is not a shift. However, it moves each point at a bounded distance. By Remark 3 we are done. $\square$

Let derive the theorem from the lemma. Let the set of Wang tiles consist of all pictures from A_3. The colour on the West side of a picture is the content of the strip of size 2×3 along the West side of the picture. Colours on other sides of Wang tiles are defined in a similar way. Tilings of a plane by these tiles correspond in a natural way to mappings B from $\mathbb{Z} \times \mathbb{Z}$ to $\{1, \ldots, k\}$ with $B_3 \subset A_3$. This correspondence respects periodicity, so the theorem is proved. $\square$

References

[1] R. Ammann, B. Grünbaum and G.C. Shephard. Aperiodic tiles. Discrete and Computational Geometry 8 (1992) 1–25.

[2] R. Berger. The undecidability of the domino problem. Memoirs of the American Mathematical Society, 66, 1966.

[3] H. Wang. Proving theorems by pattern recognition. Bell System Tech. Journal 40(1):1–41.

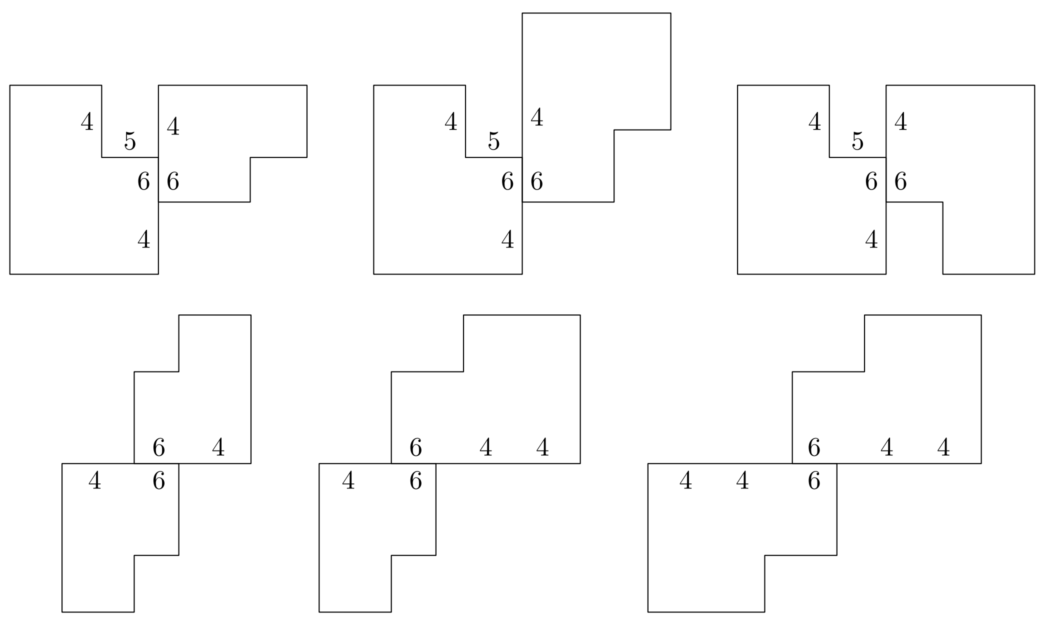
6 Appendix

6.1 Every almost proper tiling of a convex set is proper

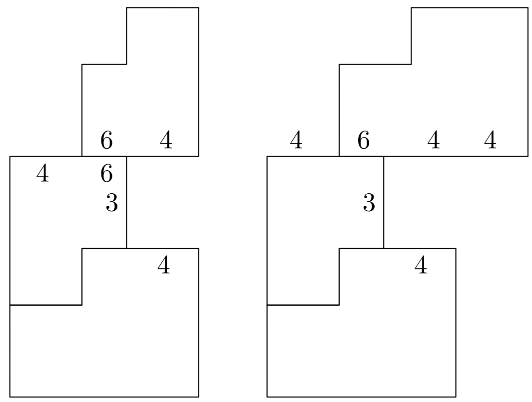
First notice that in the proof of item (b) of Lemma 4 we have not used orientation of coloured segments. This implies that every almost proper

tiling T has the coarsening T'.

Second we show that every almost proper tiling of a convex set is proper with respect to arrows $\overrightarrow{6}$: if two hexagons H, G from T share a segment $\overrightarrow{6}$ then it has the same orientation in H and G. This is obviously true if H or G is small and $\overrightarrow{6}$ lies in its cavity. Except the occurrence in the cavity of small hexagon, the arrow $\overrightarrow{6}$ occurs three times in Ammann hexagons: on the bottom of a small hexagon and on the front and back of a large hexagon. These three occurrences form six unordered pairs. So we have to show that the following six pairs cannot occur in an almost proper tiling of a convex set:

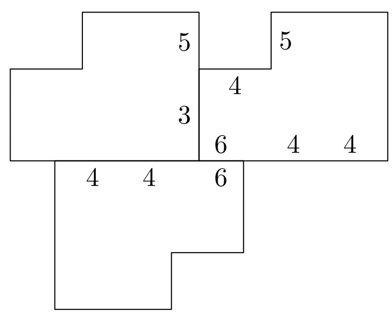


For three top pairs this is obvious. For the third and fourth pairs, the bottom small hexagon should have a brother:



In both cases in is impossible to fill in the cavity with sided coloured in 4, 3, 4.

For the sixth pair, the angle between sides 3 and 4 can be filled only by an angle of the large hexagon:



Notice that the cavity with sides 5, 4, 5 cannot be filled.

Third we show that the coarsening T' of every almost proper tiling T of a convex set is almost proper as well. Recall how we have proved a similar statement for proper tilings. We noticed that the standard colouring of T' can be obtained in three steps: (1) colouring in the standard way all hexagons in T, (2) replacing every pair (brother, sister) in T by their common parent, leaving all colours on sides as they are and (3) replacing in each hexagon in T' the colours using the reverse transformation on arrows:

$$\begin{aligned} \overrightarrow{6}\,\overrightarrow{4} &\to \overleftarrow{3} \\ \overrightarrow{5} &\to \overrightarrow{6} \\ \overrightarrow{4} &\to \overrightarrow{5} \\ \overrightarrow{3} &\to \overrightarrow{4}. \end{aligned}$$

Then we showed that the standard colouring of T has the following property: if a hexagon $H \in T$ has a pattern $\overrightarrow{6}\,\overrightarrow{4}$ and another hexagon $G \in T$ shares with H the arrow $\overrightarrow{4}$ from this pattern, then the arrow $\overrightarrow{4}$ is also preceded by $\overrightarrow{6}$ in G or that segment $\overrightarrow{6}$ is shared by the sister $K \in T$ of H. This was implied by the assumption that the shared arrow $\overrightarrow{6}$ has the same orientation in H and K (see Fig. 6). Recall that we have just proved that this assumption holds for almost proper tilings as well. Thus almost proper tilings also have this property.

Finally, apply the coarsening to a given almost proper tiling T. In the resulting almost proper tiling T_1 all adjacent segments with colour 6 have the same orientation. Then perform the refinement back. The oriented segments coloured 6 are transformed to oriented segments coloured 5. This proves that T is also proper with respect to segments coloured 5. For segments coloured 4 we need to perform two coarsenings and then two refinements, and for segments coloured 3 three coarsenings and then three refinements.

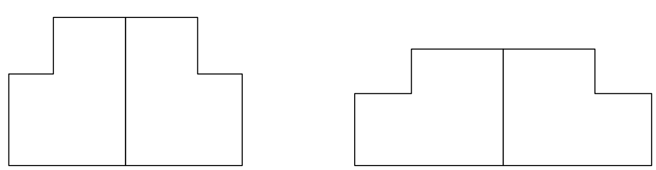
6.2 The list of allowed pairs

Recall that a pair (G, H) of Ammann hexagons is called allowed if G, H share a common segment of their boundary and the pair (G, H) appears in a standard tiling. By Theorem 1 all shared points in such G, H must the same colour and orientation.

It turns out that all allowed pairs appear in Fig. 3 (we omit the proof of this). Thus we will just list all the pairs of adjacent hexagons from there. We will divide the list in three parts, depending on whether G and H are large or small.

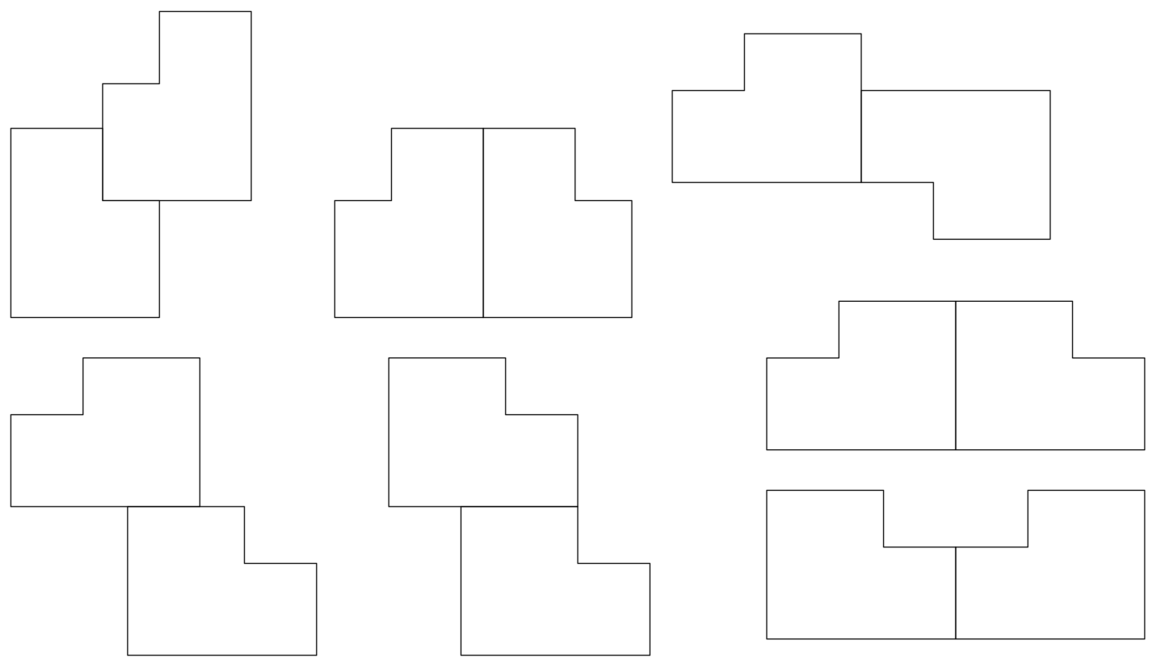
6.2.1 Both G and H are small

There are only 2 allowed pairs (out of all 6 pairs of properly attached hexagons):



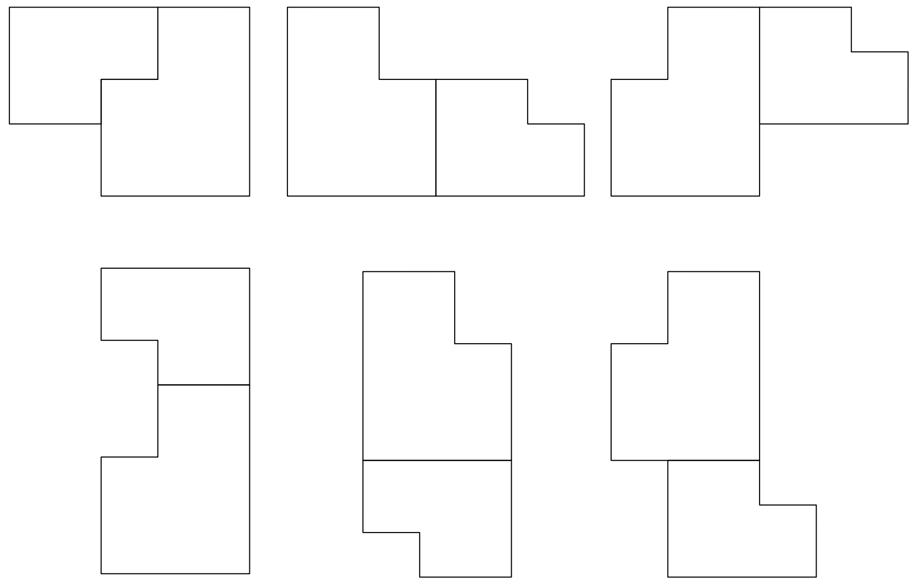
6.2.2 Both G and H are large

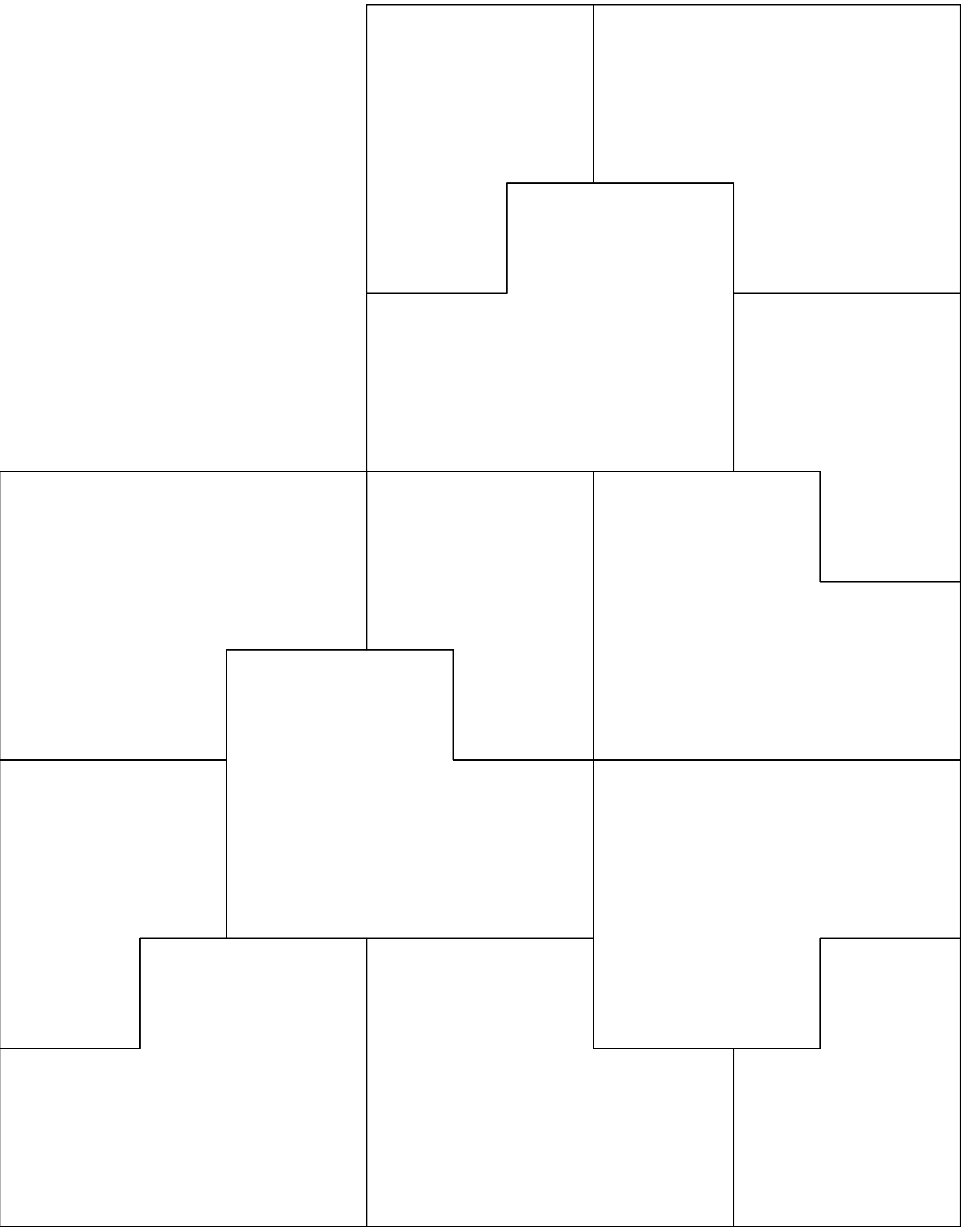
There are 7 such allowed pairs (out of all 9 pairs of properly attached hexagons):

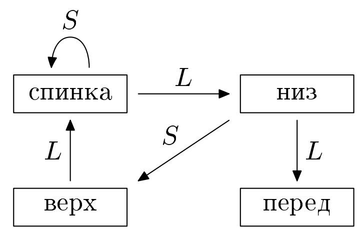


6.2.3 G is small and H is large

There are 6 such allowed pairs (out of all 12 pairs of properly attached hexagons):







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